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Conformal properties of harmonic spinors and lightlike geodesics in signature (1, 1)

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Abstract

We are studying the harmonic and twistor equation on Lorentzian surfaces, that is a two-dimensional orientable manifold with a metric of signature $(1, 1)$. We will investigate the properties of the solutions of these equations and try to relate the conformal invariant dimension of the space of harmonic and twistor spinors to the natural conformal invariants given by the Lorentzian metric. We will introduce the notion of semi-conformally flat surfaces and establish a complete classification of the possible dimensions for this family.

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1. Introduction

On every spin manifold we can canonically construct two first-order differential operators, the so-called Dirac and twistor operator. It is known that the dimension of their resp. kernels—the space of harmonic resp. twistor spinors—is a conformal invariant, that is invariant under multiplication of the metric used to define these operators with a smooth and strictly positive function. A natural question that arises is to know how these dimensions can be expressed in terms of conformal invariants given by the (pseudo-)Riemannian metric. This question is particularly interesting in the case of Riemannian and Lorentzian surfaces since they carry a natural conformal structure induced by the isothermal charts.

On Riemannian surfaces, harmonic spinors were studied in [6,4]. The dimension of the space of harmonic spinors depends essentially—unlike the dimension of the space of

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harmonic forms—on the conformal class of the metric and the spin structure used to define the Dirac operator. Furthermore, the dimension is bounded. The purpose of this paper is to study the dimension of the space of harmonic resp. twistor spinors in the case of an indefinite, that is a Lorentzian metric, and to relate them to lightlike vector fields and lightlike geodesics. We will show that:

- harmonic and twistor spinors can be described in the same way;
- harmonic and twistor spinors are linked to the global behaviour of the lightlike geodesics and the given spin structure;
- nowhere vanishing harmonic resp. twistor spinors cause conformal flatness.

Though the Lorentzian theory of surfaces looks similar to its Riemannian analogue, various new phenomena occur. For instance, we have uncountably many conformal classes of simply connected Lorentzian surfaces (see [10]). On the other hand, the torus is the only compact surface allowed to carry a Lorentzian metric since the Euler number has to vanish. However, a Lorentzian torus—unlike its Riemannian counterpart—need not be conformally flat as we have no regularity properties for solutions of hyperbolic partial differential equations. Further difficulties are caused by the non-compactness of the isometry group and the indefiniteness of the scalar product on the spinor bundle. In order to by-pass these problems, different approaches are carried out: For Lorentzian tori provided with a left-invariant metric, we can explicitly compute the kernels by tools developed in [1]. Non-conformally flat examples of Lorentzian tori are given in Section 4.2.2, where we consider a particular class of metrics for which the resulting partial differential equations with respect to the trivial spin structure can be explicitly solved. We will generalize these examples by the observation that harmonic and twistor half-spinors might be interpreted as parallel spinors along one family of lightlike geodesics. We shall introduce *semi-conformally flat* (s.c.f.) Lorentzian surfaces which are a particular class of time-orientable, non-conformally flat Lorentzian surfaces (see Definition 4.35) for which a classification of the possible dimensions in dependence on the spin structure and the global properties of the lightlike geodesics is achieved. These surfaces can by characterized by the existence of a divergence-free lightlike vector field (cf. Proposition 4.38). Furthermore, we will rederive some geometric properties of Lorentzian tori shown in [9].

We now want to state our main result. The lightcones in the tangent space induce two one-dimensional lightlike distributions which according to Section 2 may be labelled unambiguously by $\mathcal{X}$ and $\mathcal{Y}$ provided the surface is orientable (which we always tacitly assume). Furthermore, a lightlike vector field is a section either of the $\mathcal{X}$- or of the $\mathcal{Y}$-distribution. It makes therefore sense to speak of an $\mathcal{X}$- or $\mathcal{Y}$-flow if the corresponding vector fields lie in the $\mathcal{X}$- or $\mathcal{Y}$-distribution, of $\mathcal{X}$- or $\mathcal{Y}$-geodesics or $\mathcal{X}$- or of $\mathcal{Y}$-conformal flatness (depending on the divergence-free lightlike vector field to be $\mathcal{X}$ or $\mathcal{Y}$), etc. By the classical Poincaré–Bendixson theory for ordinary differential equations on a torus we know that such lightlike $\mathcal{X}$- and $\mathcal{Y}$-integral curves or "lines" are either closed, asymptotic of a closed curve or dense. These global properties of the lightlike integral curve give raise to the notion of "resonant" and "non-resonant" cylinders and tori. On the other hand the existence of harmonic spinors imposes some extra conditions on the holonomy of the principal spin bundle: If there is a closed $\mathcal{X}$- and $\mathcal{Y}$-line, then the lift of this line to the spin bundle has to be closed as well. This is what we mean by "$\mathcal{X}$"- resp. "$\mathcal{Y}$-triviality". Let $\delta_\pm = \dim \ker(D^\pm :$

$\Gamma(S^{\pm}) \to \Gamma(S^{\mp})$) resp. $\tau_{\pm} = \dim \ker(P^{\pm} : \Gamma(S^{\pm}) \to \Gamma(S^{\mp}))$ denote the dimensions of the spaces of positive/negative harmonic resp. twistor half-spinors. Then we assert the following to be true (cf. Theorem 4.48).

Theorem. *Let (M^{1+1}, g) be a compact $\mathcal{X}$-conformally flat Lorentzian surface. Then $\delta_+ = \tau_-$ and the only possible dimensions for δ_+ are $0, 1$ and $+\infty$. These cases are characterized as follows*:

(i) $\delta_+ \leq 1$ *if and only if either*

- *there exists a dense $\mathcal{X}$-line in which case we have $\delta_+ = 0$ for the non-trivial spin structures, or*
- *M^{1+1} is non-resonant, or*
- *there exists no $\mathcal{X}$-trivial resonant cylinder on M^{1+1}.*

Furthermore, we have $\delta_+ = 1$ for the trivial spin structure.

(ii) $\delta_+ = +\infty$ *if and only if there exists an $\mathcal{X}$-trivial resonant cylinder on M^{1+1}. In this case, we have $\delta_+ = +\infty$ for every spin structure.*

The same conclusion holds for $\mathcal{Y}$ and δ_- instead of $\mathcal{X}$ and δ_+, and an analogous assertion can be stated for twistor spinors.

The question to what extent this result carries over to general Lorentzian surfaces remains to be settled.

2. Lorentzian surfaces

We will give a brief introduction to the theory of Lorentzian surfaces. For details, see [10].

A Lorentzian surface (M^{1+1}, g) is given by a smooth and orientable two-dimensional manifold provided with an indefinite metric, that is TM^{1+1} splits into the direct sum of a *timelike* bundle ξ and a *spacelike* bundle η. Furthermore, the lightcone defined by g is built out of two locally integrable *lightlike* (or *isotropic*) distributions. We call these distributions $\mathcal{X}$ and $\mathcal{Y}$ according to the following convention: a vector $v \in TM^{1+1}$ lies in $\mathcal{X}$ if and only if there exists a further lightlike vector w such that (v, w) is an oriented basis and $v+w$ is spacelike. This convention is well-defined, and reversing the orientation interchanges $\mathcal{X}$ with $\mathcal{Y}$. Therefore, we can assign to any lightlike object the $\mathcal{X}$- resp. $\mathcal{Y}$-type and speak of $\mathcal{X}$- resp. $\mathcal{Y}$-vector fields, curves, geodesics, etc. We remark that lightlike vector fields need not exist globally. In fact, their global existence is equivalent to the existence of a global orthonormal basis, so that the orthonormal frame bundle over M^{1+1} is isomorphic to $M^{1+1} \times SO_+(1, 1)$ where $SO_+(1, 1)$ denotes the identity component of the isometry group $O(1, 1)$. Equivalently, we may assume the existence of a non-vanishing timelike vector field. Lorentzian surfaces which admit such vector fields are said to be *time-orientable*, since they induce an orientation in a timelike subbundle. Simply connected Lorentzian surfaces are always time-orientable. For further reference, we introduce the following notation: if we fix an othonormal basis $s = (s_1, s_2)$, let $X_s = s_1 + s_2$ and $Y_s = -s_1 + s_2$ which are $\mathcal{X}$ resp. $\mathcal{Y}$. By a suitable change of the orthonormal basis, every $\mathcal{X}$- resp. $\mathcal{Y}$-vector field can be written in this form.

The local integrability of the lightlike distributions guarantees the existence of *isotropic* resp. *isothermal* coordinates (x, y), so we may locally write $g = \lambda^2\,\mathrm{d}x\,\mathrm{d}y$ resp. $g = \lambda^2(-\mathrm{d}x^2 + \mathrm{d}y^2)$ for a smooth $\lambda \neq 0$. In particular, Lorentzian surfaces are locally conformally flat. Recall that two metrics g_1 and g_2 on a manifold M are said to be conformally equivalent if and only if there is a smooth $\lambda > 0$ such that $g_2 = \lambda g_1$. In the case where g_2 is flat, we say that g_1 is *conformally flat*. An atlas consisting of isotropic or isothermal charts defines—as in the Riemannian case—a *conformal structure* on the surface. These correspond bijectively to conformal classes of Lorentzian metrics. It should be noted, however, that the corresponding transition functions have no regularity properties. The two isotropic distributions $\mathcal{X}$ and $\mathcal{Y}$ are conformal invariants of the Lorentzian surface (M^{1+1}, g). In fact, they determine the conformal class $[g]$. The maximal integral curves of the $\mathcal{X}$- and $\mathcal{Y}$-vector fields which we call $\mathcal{X}$- and $\mathcal{Y}$-lines or *null lines* for short, are further conformal invariants. The $\mathcal{X}$- and $\mathcal{Y}$-lines through x will be denoted by l_x and m_x. As we can locally straighten out the null lines by choosing isotropic null coordinates, only the global properties of the null lines encode conformal information. In the case of a simply connected surface, it can be shown that there are no closed null lines, that two different null lines intersect in at most one point and that every null line is properly embedded in M^{1+1}.

Since the existence of a Lorentzian metric on a compact surface is equivalent to $\chi(M) = 0$ where χ denotes the Euler characteristic, every compact Lorentzian surface is diffeomorphic to a torus. According to the Poincaré–Bendixson theory for ordinary differential equations on the torus, a null line on a compact Lorentzian surface is either dense, a closed curve homeomorphic to S^1 which cannot be contracted to a point, or an asymptotic of a closed null line of the same type.

In [9], the explicit behaviour of the null lines and further properties are discussed for metrics of the form

$$g_{(x_1,x_2)} = E(x_1)\,\mathrm{d}x_1^2 + 2F(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 - G(x_1)\,\mathrm{d}x_2^2.$$

Up to a finite covering, all Lorentzian tori with non-trivial isometry group are of that type. If $G \equiv 0$ resp. $|G| > 0$, then g is flat resp. conformally flat. We consider the family of metrics $\mathcal{G}'$ where $G(0) = 0$ and G has only isolated zeros in $p_0 = 0, p_1, \ldots, p_{n-1} \in (0, 1)$, $p_{n+k} = p_k + 1$ for all integer k. Then $(\mathbb{R}^{1+1}, g)$ is incomplete in the three causal senses for all $g \in \mathcal{G}'$, and so is (T^{1+1}, g).

In particular, let us consider the two subfamilies

$$\mathcal{G}_1 = \{g \in \mathcal{G}' | G_{|(0,1)} > 0\}, \tag{2.1}$$

$$\mathcal{G}_2 = \{g \in \mathcal{G}' | G'(p_i) \neq 0,\ F(p_i)F(0) > 0,\ 0 \leq i \leq n-1\}. \tag{2.2}$$

We have the following proposition (see [9]).

Proposition 2.1. *Let* $g \in \mathcal{G}_1 \cup \mathcal{G}_2$.

1. *Let* $\eta_0 = \operatorname{sgn} F(0)$. *Then*

$$X_1 = G\partial_{x_1} + (F + \eta_0\sqrt{EG + F^2})\partial_{x_2}, \quad X_2 = \partial_{x_1} + \left(\frac{F - \eta_0\sqrt{EG + F^2}}{G}\right)\partial_{x_2}$$

are two linearly independent isotropic vector fields for g (*the choice of* η_0 *guaranteeing the existence of the limit in* p_i).

2. *The inextendible null geodesics of* X_2 *are complete. Hence there exists incomplete* X_1*-geodesics since* (T^{1+1}, g) *is lightlike incomplete*.

As we remarked above, a Lorentzian torus need not be conformally flat. In fact, we have the following characterization of conformal flatness (see [9]).

Proposition 2.2. *Let* (M^{1+1}, g) *be a Lorentzian surface*.

1. *If there is a nowhere vanishing time- or spacelike conformal vector field, then* (M^{1+1}, g) *is conformally flat. The converse is true if in addition M is compact*.
2. *Every conformally flat compact Lorentzian surface is complete*.

We recall that a vector field K is called *conformal* if $\mathcal{L}_K g = \sigma g$ for a smooth function σ (where $\mathcal{L}$ denotes the Lie derivative).

3. Pseudo-Riemannian spin geometry

We will give a brief survey of the relevant spin geometric features we use in the fourth section. We focus mainly on the signature (1, 1). A general reference is [1].

Let $(\mathbb{R}^{p+q}, \langle\cdot,\cdot\rangle_{p,q})$ be the standard pseudo-Euclidean vector space of signature (p, q) where p is the dimension of a maximal timelike subspace. We shall always assume that the p first vectors of an orthonormal basis are timelike. For $p + q = 2m$ we can identify the associated clifford algebra $Cl_{p,q} = \mathrm{Cliff}(\mathbb{R}^{p+q}, \langle\cdot,\cdot\rangle_{p,q})$ with $\mathrm{End}(\Delta_{p,q}) = \mathbb{C}(2^m)$ obtaining thereby an action μ of $Cl_{p,q}$ on $\Delta_{p,q} = \mathbb{C}^{2^m}$. This action will be denoted by $\cdot$, that is $\mu(x, v) = x \cdot v$. An explicit isomorphism in signature (1, 1) is given by extension of the mapping

$$e_1 \mapsto \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad e_2 \mapsto \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{3.1}$$

Next, we define the groups $\mathrm{Spin}(p, q)$ and $\mathrm{Spin}_+(p, q)$. Let $S_{p,q} = \{x \in \mathbb{R}^{p+q} | \langle x, x\rangle_{p,q} = 1\}$ and $H_{p,q} = \{x \in \mathbb{R}^{p+q} | \langle x, x\rangle_{p,q} = -1\}$.

Definition 3.1.

$$\mathrm{Spin}(p, q) = \{x_1 \cdots x_{2l} | x_i \in H_{p,q} \cup S_{p,q}\},$$
$$\mathrm{Spin}_+(p, q) = \{x_1 \cdots x_{2l} | x_i \in H_{p,q} \cup S_{p,q} \ \text{ with an even number of timelike factors}\}.$$

In order to deal simultaneously with the pairs $\mathrm{SO}(p, q)/\mathrm{Spin}(p, q)$ and $\mathrm{SO}_+(p, q)/\mathrm{Spin}_+(p, q)$, we write $G(p, q)$ and $\tilde{G}(p, q)$, where $\tilde{G}(p, q) = \mathrm{Spin}(p, q)$ if $G(p, q) = \mathrm{SO}(p, q)$ and $\tilde{G}(p, q) = \mathrm{Spin}_+(p, q)$ if $G(p, q) = \mathrm{SO}_+(p, q)$. If we use (3.1) to represent $\mathrm{Spin}(1, 1)$, we get the following lemma.

Lemma 3.2.

$$\mathrm{Spin}(1,1) = \left\{ g_a = \begin{pmatrix} a & 0 \\ 0 & \pm\dfrac{1}{a} \end{pmatrix} | a \in \mathbb{R}\setminus\{0\} \right\}.$$

The volume element $\omega = e_1 \cdot e_2$ of $Cl_{1,1}$ defines—viewed as an endomorphism of $\Delta_{1,1}$—a splitting of $\Delta_{1,1}$ into the direct sum of $\Delta^+_{1,1}$ = Eigenspace of ω for $-1 = \langle z_1 \rangle$ and $\Delta^-_{1,1}$ = Eigenspace of ω for $1 = \langle \mathrm{i} z_2 \rangle$, where (z_1, z_2) denotes the standard basis of $\mathbb{C}^2$ (the sign convention follows [1] and is motivated by the higher dimensional case). Reversing the orientation interchanges $\Delta^+_{1,1}$ with $\Delta^-_{1,1}$.

Next, we consider spin structures of (M^{p+q}, g), that is reductions (Q, f) of the $G(p,q)$-frame bundle P to a $\tilde{G}(p,q)$-bundle Q. We have the following criterion for the existence of such reductions.

Proposition 3.3. *Let (M^{p+q}, g) be a connected pseudo-Riemannian manifold and $TM = \xi^p \oplus \eta^q$ a splitting into a time- and spacelike bundle resp. of maximal rank.*

(i) *(M^{p+q}, g) is spin if and only if $w_2(TM) = w_1^2(\eta)$, where $w_i \in H^i(M, \mathbb{Z}_2)$ denotes the ith Stiefel–Whitney-class.*
(ii) *If (M^{p+q}, g) is time-oriented, then the mapping* $\mathrm{Spin}(M^{p+q}, g) \to \pi_1(P, x)$, $(Q, f) \mapsto f_* \pi_1(Q, y)$ *for $y \in f^{-1}(x)$ is injective. In particular, two spin structures which are isomorphic as a twofold covering of P are isomorphic as spin structures.*
(iii) *If* $\mathrm{Spin}(M^{p+q}, g) \neq \emptyset$, *then* $\mathrm{card}(\mathrm{Spin}(M^{p+q}, g)) = \mathrm{card}(H^1(M^{p+q}, \mathbb{Z}_2))$.

For a proof of (i) see [7], for (ii) and (iii) see [1].

In particular, every time-orientable Lorentzian surface admits a spin structure. One is explicitly given by $Q_0 = M^{1+1} \times \mathrm{Spin}_+(1,1)$ and $f_0(x, a) = (x, \lambda(a))$. This spin structure will be referred to as the *trivial* one; it is unique up to isomorphism if M is simply connected. On the other hand, if M is time-orientable and compact, then (M^{1+1}, g) carries four non-isomorphic spin structures since $H^1(T^{1+1}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

If $\tilde{g} = \kappa^2 g$ is conformally equivalent to g, we can canonically associate a spin structure $(\tilde{Q}, \tilde{f})$ with every spin structure (Q, f) on (M^{p+q}, g): if $\Phi_\kappa : P_g \to P_{\tilde{g}}$ is the isomorphism defined by $s = (s_1, \ldots, s_{p+q}) \mapsto (1/\kappa)s = ((1/\kappa)s_1, \ldots, (1/\kappa)s_{p+q})$, then the subgroup $(\Phi_\kappa \circ f)_\# \pi_1(Q, q)$ in $\pi_1(P_{\tilde{g}}, \Phi_\kappa(f(q)))$ distinguishes by Proposition 3.3 a spin structure $(\tilde{Q}, \tilde{f})$ which can be shown to be isomorphic with (Q, f).

Now, we fix a spin structure (Q, f) over (M^{p+q}, g). The associated fibre bundle

$$S = Q \times_{\tilde{G}(p,q)} \Delta_{p,q}$$

is a complex vector bundle of rank $2^{[(p+q)/2]}$ which is called the *spinor bundle* associated with (Q, f). The set of smooth sections of S is denoted by $\Gamma(S)$: its elements are called *spinor fields* or *spinors* for short. A spinor φ may be represented by a $\tilde{G}(p,q)$-equivariant function $\tilde{\varphi} \in C^\infty(Q, \Delta_{p,q})^{\tilde{G}(p,q)}$, that is, $\tilde{\varphi}(qg) = g^{-1}\tilde{\varphi}(q)$ for all $q \in Q$ and $g \in \tilde{G}(p,q)$. One may also think of a spinor as a collection of local sections $\tilde{s} : U \to Q$ covering M together with a family of local trivializations $\varphi_{\tilde{s}} \in C^\infty(U, \Delta_{p,q})$ verifying $\varphi_{\tilde{s}g} = g^{-1}\varphi_{\tilde{s}}$.

Furthermore, we can consider the sections of $S^{\pm} = Q \times_{\tilde{G}(1,1)} \Delta^{\pm}_{1,1}$, where the fibrewise splitting is induced by the volume element of $\mathrm{Cliff}(T_x M^{1+1}, g_x)$, the Clifford algebra generated by $(T_x M^{1+1}, g_x)$. The corresponding sections in $\Gamma(S^{\pm})$ are called *half-spinors*. To emphasize the sign, we also speak of *positive* or *negative* (half-)spinors. In the subsequent chapters, we will also use the following representation of half-spinors: if $\varphi \in \Gamma(S^{\pm})$ is a positive resp. a negative spinor, we can write $\tilde{\varphi}(g) = \tilde{f}^{\pm}(q)u_{\pm 1}$ for $\tilde{f}^{\pm} \in C^{\infty}(Q, \mathbb{C})$. Using the representation of Spin(1, 1) of Lemma 3.2, the transformation law of $\tilde{f}^{+}$ is given by $\tilde{f}^{+}(qg_a) = (1/a)\tilde{f}^{+}(q)$ and $\tilde{f}^{-}(qg_a) = a\tilde{f}^{-}(q)$. The same holds if one considers the complex-valued functions $f_{\tilde{s}}$ given by the local trivializations $\varphi_{\tilde{s}} = f_{\tilde{s}}u_{\pm}$.

As for Riemannian spin bundles the covariant derivative $\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S)$ is induced by the lift to Q of the Levi–Civita connection Z in P. Fixing a local orthonormal basis $s = (s_1, s_2)$, we get

$$\nabla^S_V \varphi = [\tilde{s}, V(\varphi_{\tilde{s}}) - \tfrac{1}{2} g(\nabla^{\mathrm{LC}}_V s_1, s_2) e_1 \cdot e_2 \cdot \varphi_{\tilde{s}}]. \tag{3.2}$$

We also verify the product rule

$$\nabla^S_V (W \cdot \varphi) = (\nabla^{\mathrm{LC}}_V W) \cdot \varphi + W \cdot \nabla^S_V \varphi.$$

The main difficulty in the pseudo-Riemannian setup is to define a suitable scalar product on S. This can be done as follows: assume (M^{1+1}, g) to be time-orientable. Let $TM^{1+1} = \xi^1 \oplus \eta^1$ be a splitting into a (now trivial) time- resp. spacelike vector bundle. Fix orientations in ξ and η. P can be reduced to the structure group $K = \mathrm{SO}(1) \times \mathrm{SO}(1)$, which is maximal compact in $\mathrm{SO}_+(1, 1)$. The reduced bundle is given by $P_\xi = \{(s_1, s_2) | s_1 \text{ positively oriented in } \xi, s_2 \text{ positively oriented in } \eta\}$. Then $\tilde{Q}_\xi = f^{-1}(P_\xi)$ is the reduction of Q to $\tilde{K} = (\mathrm{Spin}_+(1) \times \mathrm{Spin}_+(1))/\mathbb{Z}_2$ which is maximal compact in $\mathrm{Spin}_+(1, 1)$. We have $S = \tilde{Q}_\xi \times_{\tilde{K}} \Delta_{1,1}$ and $TM = P_\xi \times_K \mathbb{R}^{1+1}$. Let $(\cdot, \cdot)_{\Delta_{1,1}}$ denote the standard hermitian product on $\Delta_{1,1}$ which is $\tilde{K}$-invariant, but not $\mathrm{Spin}_+(1, 1)$-invariant. We can extend this scalar product to a fibrewise defined scalar product $(\cdot, \cdot)_\xi$ on S. Now let $J_\xi : S \to S$, $J_\xi([\tilde{q}, v]) = [\tilde{q}, e_1 \cdot v]$ for $\tilde{q} \in \tilde{Q}_\xi$ and the unit vector field $e_1 \in \xi$. We define

$$\langle \varphi, \psi \rangle_x = (J_\xi \varphi, \psi)_{\xi x} = (e_1 \cdot v, w)_{\Delta_{1,1}},$$

where $\varphi(x) = [\tilde{q}, v]$ and $\psi(x) = [\tilde{q}, w]$. This is an indefinite, $\mathrm{Spin}_+(1, 1)$-invariant scalar product on S. Then the formulae

$$V(\langle \varphi, \psi \rangle) = \langle \nabla^S_V \varphi, \psi \rangle + \langle \varphi, \nabla^S_V \psi \rangle$$

and

$$\langle X \cdot \varphi, \psi \rangle = \langle \varphi, X \cdot \psi \rangle$$

hold.

Given (S, ∇^S) over (M^{1+1}, g), we can canonically define two first-order differential operators, namely the Dirac operator D and the twistor operator P. In terms of a local orthonormal basis $s = (s_1, s_2)$, they are given by

$$D\varphi = -s_1 \cdot \nabla^S_{s_1} \varphi + s_2 \cdot \nabla^S_{s_2} \varphi \quad \text{and}$$
$$P\varphi = -s_1 \otimes (\nabla^S_{s_1} \varphi + \tfrac{1}{2} s_1 \cdot D\varphi) + s_2 \otimes (\nabla^S_{s_2} \varphi + \tfrac{1}{2} s_2 \cdot D\varphi).$$

In the fourth section, we will deal with the equations $D\varphi = 0$ and $P\psi = 0$ called the *harmonic* resp. the *twistor* equation, the latter being equivalent to $\nabla^S_V\varphi = -\frac{1}{2}V \cdot D\varphi$ for every $V \in \mathfrak{X}(M^{1+1})$. The solutions are referred to as *harmonic* resp. *twistor* spinors. The vector spaces of harmonic- resp. twistor spinors will be denoted by $\mathfrak{H}$ and $\mathfrak{T}$. Furthermore, we will consider $\mathfrak{H}_\pm = \Gamma(S^\pm) \cap \mathfrak{H}$ and analogously $\mathfrak{T}_\pm = \Gamma(S^\pm) \cap \mathfrak{T}$. The superscript 0 denotes the space of harmonic and twistor spinors resp. their dimension with respect to the trivial spin structure. We are mainly interested in the numbers $\delta_{(\pm)} = \dim(\mathfrak{H}_{(\pm)})$ and $\tau_{(\pm)} = \dim(\mathfrak{T}_{(\pm)})$, since they have the following well-known property (see [2,3]).

Proposition 3.4. *Let $\tilde{g} = \lambda g$, and $\tilde{\mathfrak{H}}_{(\pm)}$ resp. $\tilde{\mathfrak{T}}_{(\pm)}$ the space of harmonic resp. twistor (half-)spinors with respect to $\tilde{g}$. Then the maps*

(i) $\tilde{\varphi} \in \tilde{\mathfrak{H}}_{(\pm)} \mapsto \lambda^{1/4}\tilde{\varphi} \in \mathfrak{H}_{(\pm)}$,
(ii) $\tilde{\psi} \in \tilde{\mathfrak{T}}_{(\pm)} \mapsto \lambda^{-1/4}\tilde{\varphi} \in \mathfrak{T}_{(\pm)}$

are isomorphisms. In particular, $\delta_{(\pm)}$ and $\tau_{(\pm)}$ are conformal invariants.

4. Spinor field equations and lightlike geodesics in signature (1,1)

4.1. Harmonic and twistor spinors

Proposition 4.1.

(i) *Let φ be in $\Gamma(S^+)$ resp. $\Gamma(S^-)$. Then φ is harmonic if and only if*

$$\nabla^S_X\varphi \equiv 0 \ \text{resp.} \ \nabla^S_Y\varphi \equiv 0$$

holds for all $\mathcal{X}$ vector fields X resp. $\mathcal{Y}$-vector fields Y.

(ii) *Let φ be in $\Gamma(S^+)$ resp. $\Gamma(S^-)$. Then φ is twistor if and only if*

$$\nabla^S_Y\varphi \equiv 0 \ \text{resp.} \ \nabla^S_X\varphi \equiv 0$$

holds for all $\mathcal{Y}$-vector fields Y resp. $\mathcal{X}$-vector fields X.

Proof. We prove the assertion only for positive spinors, the remaining cases being showed in the same way.

Let $\varphi \in \Gamma(S^+)$ and let X and Y be a $\mathcal{X}$-resp. $\mathcal{Y}$-vector field which we write $X = X_s = (s_1 + s_2)$ and $Y = Y_s = (-s_1 + s_2)$.

(i) Using the local expression of D, we see that φ is harmonic if and only if $s_1 \cdot \nabla^S_{s_1}\varphi = s_2 \cdot \nabla^S_{s_2}\varphi$ which is equivalent to $\nabla^S_{s_1}\varphi = \omega \cdot \nabla^S_{s_2}\varphi = -\nabla^S_{s_2}\varphi$ (with the volume element $\omega = s_1 \cdot s_2$). Hence φ is harmonic if and only if $\nabla^S_{s_1+s_2}\varphi = 0$.

(ii) $\nabla^S_{s_i}\varphi = -(1/2)s_i \cdot D\varphi$ for $i = 1, 2$ is equivalent to $\nabla^S_{s_1}\varphi = -\omega \cdot \nabla^S_{s_2}\varphi$ and $\nabla^S_{s_2} = -\omega \cdot \nabla^S_{s_1}\varphi$, hence to $\nabla^S_{-s_1+s_2}\varphi = 0$. □

Since $\nabla^S_V\varphi(x) = [q_x, V^*(\tilde\varphi)(q_x)]$ for all $q_x \in \pi_Q^{-1}(x)$ and $V \in \mathcal{X}(M)$ (where V^* denotes the horizontal lift of V to Q), proposition Proposition 4.1 may be restated as follows.

Corollary 4.2.

(i) $\mathfrak{H}_+ = \{\tilde\varphi \in C^\infty(Q, \Delta^+_{1,1})^{\tilde G(1,1)} | \tilde\varphi$ *is constant along the horizontal lifts of* $\mathcal{X}$*-curves*$\}$.

(ii) $\mathfrak{H}_- = \{\tilde\varphi \in C^\infty(Q, \Delta^-_{1,1})^{\tilde G(1,1)} | \tilde\varphi$ *is constant along the horizontal lifts of* $\mathcal{Y}$*-curves*$\}$.

(iii) $\mathfrak{T}_+ = \{\tilde\varphi \in C^\infty(Q, \Delta^+_{1,1})^{\tilde G(1,1)} | \tilde\varphi$ *is constant along the horizontal lifts of* $\mathcal{Y}$*-curves*$\}$.

(iv) $\mathfrak{T}_- = \{\tilde\varphi \in C^\infty(Q, \Delta^-_{1,1})^{\tilde G(1,1)} | \tilde\varphi$ *is constant along the horizontal lifts of* $\mathcal{X}$*-curves*$\}$.

A further characterization is given by the formula $\nabla^S_{\gamma'(t)}\varphi = \mathrm{d}/\mathrm{d}s\,\mathcal{P}^Q_{\gamma:t+s\to t}\varphi(\alpha(t + s))_{|s=0}$ for any smooth curve γ, where $\mathcal{P}^Q_{\gamma:t+s\to t}$ denotes the parallel transport of Q along γ between the fibres $\pi_Q^{-1}(\gamma(t+s))$ and $\pi_Q^{-1}(\gamma(t))$.

Corollary 4.3. *Let $\varphi \in \Gamma(S^+)$. Then φ is a positive harmonic spinor if and only if for any $\mathcal{X}$-curve α joining two points x and y in M^{1+1}, we have $\varphi(y) = [\mathcal{P}^Q_{\alpha:x\to y}q, v]$ for $\varphi(x) = [q, v]$. Analogous statements hold for $\mathfrak{H}_-$, $\mathfrak{T}_+$ and $\mathfrak{T}_-$.*

As a first application, we note the following proposition.

Proposition 4.4. *There is a bijective correspondence between the sets $\{\varphi \in \mathfrak{H}_+ | \varphi(x) \neq 0$ for all $x\}$ and $\{\varphi \in \mathfrak{T}_- | \psi(x) \neq 0$ for all $x\}$.*

Proof. If $\varphi \in \Gamma(S^+)$ is given by $\tilde f^+ u_1$ for $\tilde f^+ \in C^\infty(Q, \mathbb{C})$, we can define a twistor spinor $\psi_{\tilde f^+} \in \Gamma(S^-)$ by $\tilde\psi_{\tilde f^+} = 1/\tilde f^+ u_{-1}$, since $\psi_{\tilde f^+}(qg_a) = 1/\tilde f^+(qg_a) = a(1/\tilde f^+(q)) = a\psi_{\tilde f^+}(q) = g_a^{-1}\psi_{\tilde f^+}(q)$, so $1/\tilde f^+ u_{-1}$ defines indeed a $\tilde G(1,1)$-invariant function. Because of $X^*(1/\tilde f^+) = 0$, $\psi = [q, 1/\tilde f^+(q)u_{-1}]$ defines a negative twistor spinor. □

Proposition 4.5. *Let $\varphi \in \mathfrak{H}_+$. If $\varphi(x) = 0$, then $\varphi_{|l_x} \equiv 0$. In particular, we have $\delta_+ \leq 1$ for any spin structure if there exists a dense null line on (M^{1+1}, g). Analogous statements hold for $\mathfrak{H}_-$, $\mathfrak{T}_+$ and $\mathfrak{T}_-$.*

Proof. The first assertion is a consequence of the above corollaries. Assume that there is an $x \in M^{1+1}$ with l_x is dense in M^{1+1}. Let $\varphi_1, \varphi_2 \in \mathfrak{H}_+$ with $\varphi_1 \not\equiv 0$. Pick $c \in \mathbb{C}$ such that $\varphi_2(x) = c\varphi_1(x)$. Hence $(\varphi_2 - c\varphi_1)_{|l_x} \equiv 0$, that is $\varphi_2 \equiv c\varphi_1$ for continuity reasons. □

4.2. Examples

We now apply the preceding results to compute some explicit examples. For the sake of simplicity, we will only deal with positive harmonic spinors, but all examples extend to the remaining cases in an obvious way.

4.2.1. Simply connected surfaces

As observed in Section 2, two different null lines intersect at most once and closed null lines cannot exist. Furthermore, the frame bundle P is trivial since (M^{1+1}, g) is time-orientable, and the resulting trivial spin structure is unique up to isomorphism.

Proposition 4.6. *If M^{1+1} is simply connected, then $\delta_+ = +\infty$.*

Proof. Let $\beta : [0, 1] \to M^{1+1}$ be a $\mathcal{Y}$-curve with lift $\tilde{\beta}$ to Q, and let $f_n : [0, 1] \to \mathbb{C}$ be a family of linearly independent smooth functions whose support is strictly contained in $[0, 1]$. Define a $\mathrm{Spin}_+(1, 1)$-equivariant function $\tilde{\varphi}_n : Q \to \Delta^+_{1,1}$ by extending $\tilde{\varphi}_n(\tilde{\beta}(t)) = f_n(t)u_1$ first to $Q_{|\beta}$ by the transitive action of $\mathrm{Spin}_+(1, 1)$ on the fibres, and secondly to M^{1+1} by parallel transport on $A = \bigcup_{x\in\beta} l_x$ and $\tilde{\varphi}_n \equiv 0$ on A^c. □

This construction depends crucially on the fact that for simply connected surfaces, the local and the global behaviour of the null lines are the same. Therefore, we can extend local solutions to global ones. This observation is the key for the construction of harmonic and twistor spinors in Section 4.3: we will link the global behaviour of the null lines to the spinors; by studying the null lines in the large, we will be able to extend local solutions or to find obstructions for doing so.

4.2.2. Diagonal metrics on Lorentzian tori

We consider Lorentzian tori (T^{1+1}, g_λ) whose metric is given by a *diagonal metric*

$$g_\lambda(x_1, x_2) = -\lambda_1^2(x_1, x_2)\mathrm{d}x_1^2 + \lambda_2^2(x_1, x_2)\,\mathrm{d}x_2^2$$

for $\lambda_1, \lambda_2 \neq 0$ in $C^\infty(\mathbb{R}^2)^{\mathbb{Z}^2}$.

4.2.2.1. Left-invariant metrics. First we consider the case where λ_1, λ_2 are constant. Thus (T^{1+1}, g_λ) may be seen as a Lie group provided with a left-invariant metric. Since $\pi_1(T^{1+1})$ has no two-torsions, we may treat the harmonic and the twistor equation for all four spin structures simultaneously by tools developed in [1] which we will briefly sketch.

The problem is to compare two non-isomorphic spin structures (Q_1, f_1) and (Q_2, f_2) and to find conditions for $\Gamma(S_1)$ and $\Gamma(S_2)$ to be isomorphic. Let (M^{p+q}, g) be a pseudo-Riemannian spin manifold of signature (p, q). Let $R' = \{(q_1, q_2) \in Q_1 \times Q_2 | f_1(q_1) = f_2(q_2)\}$. $\mathbb{Z}_2$ acts naturally on each fibre of Q_i, hence on R'. The pair (R, μ), where $R = R'/\mathbb{Z}_2$ and $\mu : R \to P, [q_1, q_2] \mapsto f_1(q_1)$, is called the *deformation* of (Q_1, f_1) and (Q_2, f_2). If (Q_1, f_1) and (Q_2, f_2) are isomorphic, then R is isomrphic to $P \times \mathbb{Z}_2$. $G(p, q) \times \mathbb{Z}_2$ acts on R by $[q_1, q_2], (A, m) = [q_1a, q_2am]$, where $a \in \lambda^{-1}(A)$. This action is well defined, therefore providing R with the structure of a $G(p, q) \times \mathbb{Z}_2$-fiber bundle. Next we define the vector bundle $E = R/G(p, q) \times_{\mathbb{Z}_2} \mathbb{R}$ over M. Its complexification $E^{\mathbb{C}}$ is given by $E^{\mathbb{C}} = R/G(p, q) \times_{\mathbb{Z}_2} \mathbb{C}$. Let $\tilde{s}_i : U \to Q_i$ be two local sections and let $[\tilde{s}] = [(\tilde{s}_1, \tilde{s}_2)] : U \to R$, where $[\cdot]$ denotes the equivalence classes in R. Let $e \in E^{\mathbb{C}}$. Then e can be represented in the form $e = [\{\tilde{s}\}, z]$ ($\{\cdot\}$ denoting the equivalence classes in $R/G(p, q)$).

Proposition 4.7. *The map* $\beta : S_1 \otimes E^{\mathbb{C}} \to S_2$ *defined by* $\beta([\tilde{s}_1, v]_x \otimes [\{\tilde{s}\}, z]_x) = [\tilde{s}_2, zv]_x$ *is a vector bundle isomorphism.*

Hence, in the case where $E^{\mathbb{C}}$ is trivial, the spinor bundles S_1 and S_2 are isomorphic. For instance, this happens if (Q_1, f_1) and (Q_2, f_2) are isomorphic, for $E^{\mathbb{C}}$ is then isomorphic to $M^{p+q} \times \mathbb{Z}^2$. Thus equivalent spin structures induce isomorphic spinor bundles. Furthermore, we yield the following corollary.

Corollary 4.8. *On a surface M each two spinor bundles are isomorphic.*

Proof. For the first Chern class of the complexification $E^{\mathbb{C}}$ of the real line bundle E holds $2c_1(E^{\mathbb{C}}) = 0$. Since $H^2(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$ depending on whether or not M is compact, $E^{\mathbb{C}}$ must be trivial. □

Next we want to know how the spinor derivative transforms under this isomorphism.

Let $\nabla^{E^{\mathbb{C}}}$ be the connection induced by the lift to R of the Levi–Civita connection of P. Then one shows that $\nabla^{E^{\mathbb{C}}}$ is flat, hence for $\eta = [\{\tilde{s}\}, z] \in \Gamma(E^{\mathbb{C}}_{|U})$ with $z : U \to \mathbb{C}$, we have $\nabla^{E^{\mathbb{C}}}_V \eta = [\{\tilde{s}\}, V(\eta)]$. Therefore, the following diagram commutes for every $V \in \mathcal{X}(M^{p+q})$:

$$\begin{array}{ccc}
\Gamma(S_1 \otimes E^{\mathbb{C}}) & \xrightarrow{\beta} & \Gamma(S_2) \\
\Big\downarrow \nabla^{S^1}_V \otimes_{\nabla^{E^{\mathbb{C}}}_V} 1 & \circlearrowleft & \Big\downarrow \nabla^{S_2}_V \\
\Gamma(S_1 \otimes E^{\mathbb{C}}) & \xrightarrow{\beta} & \Gamma(S_2)
\end{array}$$

where $\nabla^{S_1}_V \otimes_{\nabla^{E^{\mathbb{C}}}_V} 1(\varphi \otimes \eta) = \nabla^{S_1}_V \varphi \otimes \eta + \varphi \otimes \nabla^{E^{\mathbb{C}}}_V \eta$.

If we assume $E^{\mathbb{C}}$ to be trivial, then we can choose a nowhere vanishing section $e : M^{p+q} \to E^{\mathbb{C}}$. We define the complex-valued form ω_e by the equation $\nabla^{E^{\mathbb{C}}}_V e = \omega_e(V)e$ and $\alpha_e : \Gamma(S_1) \to \Gamma(S_1 \otimes E^{\mathbb{C}})$ by $\alpha_e(\varphi)(x) = \varphi_x \otimes e_x$. Then the following diagram commutes:

$$\begin{array}{ccccc}
\Gamma(S_1) & \xrightarrow{\alpha_e} & \Gamma(S_1 \otimes E^{\mathbb{C}}) & \xrightarrow{\beta} & \Gamma(S_2) \\
\Big\downarrow \nabla^{S^1}_V + \omega_e(V) & \circlearrowleft & \Big\downarrow \nabla^{S}_V \otimes_{\nabla^{E^{\mathbb{C}}}_V} 1 & \circlearrowleft & \Big\downarrow \nabla^{S_2}_V \\
\Gamma(S_1) & \xrightarrow{\alpha_e} & \Gamma(S_1 \otimes E^{\mathbb{C}}) & \xrightarrow{\beta} & \Gamma(S_2)
\end{array}$$

Let us now consider the special case of a connected Lie group G provided with a left-invariant metric g. Let $p : \tilde{G} \to G$ be the universal cover of G. $\pi_1(G)$ acts as a group of deck transformations. Since g is left-invariant, we can trivialize P by choosing n left-invariant vector fields on G, that is $P = G \times \mathrm{SO}_+(p, q)$. Therefore, $Q_0 = G \times \mathrm{Spin}_+(p, q)$ with $f_0 = id \times \lambda$ defines the trivial spin structure on G. The lifts $\tilde{X}_i$ of the vector fields X_i to $\tilde{G}$ are globally left- and $\pi_1(G)$-invariant vector fields on $\tilde{G}$, hence $\tilde{P} = \tilde{G} \times \mathrm{SO}_+(p, q)$

and $\tilde{Q}_0 = \tilde{G} \times \mathrm{Spin}_+(p,q)$, $\tilde{f}_0 = id \times \lambda$. We know that $\mathrm{Spin}(G,g) \cong \mathrm{Hom}(\pi_1(G), \mathbb{Z}_2)$. Let $\chi \in \mathrm{Hom}(\pi_1(G), \mathbb{Z}_2)$. $\pi_1(G,g)$ acts on $\tilde{G} \times \mathrm{Spin}_+(p,q)$ through χ by $\omega, (\tilde{g}, a) = (\omega, \tilde{g}, \chi(\omega)a)$, where $\omega \in \pi_1(G,e)$. Let $Q_\chi = \tilde{G} \times_{[\pi_1(G,e),\chi]} \mathrm{Spin}_+(p,q)$ and $f_\chi : Q_\chi \to P$, $[\tilde{g}, a] \mapsto [p(\tilde{g}), \lambda(a)]$. Then (Q_χ, f_χ) defines a spin structure and the following propostion holds (cf. [1]).

Proposition 4.9.

(i) $\mathrm{Spin}(G,g) \cong \{(Q_\chi, f_\chi) | \chi \in \mathrm{Hom}(\pi_1(G), \mathbb{Z}_2)\}$.
(ii) *The spinor bundle* $S_\chi = Q_\chi \times_{\mathrm{Spin}_+(p,q)} \Delta_{p,q}$ *associated with the spin structure* (Q_χ, f_χ) *is given by* $S_\chi = \tilde{G} \times_\chi \Delta_{p,q}$.
(iii) *The deformation of* (Q_0, f_0) *and* (Q_χ, f_χ) *is given by* $R_\chi = (\tilde{G}/\ker(\chi)) \times \mathrm{SO}_+(p,q)$. *Furthermore,* $E^{\mathbb{C}}_\chi = (\tilde{G}/\ker(\chi)) \times_{\mathbb{Z}_2} \mathbb{C}$.

Assume that we have a nowhere vanishing section $e_\chi \in \Gamma(E^{\mathbb{C}})$. Such a section is given by a map $\epsilon_\chi : \tilde{G} \to \mathbb{C}$ without zeros such that $\epsilon(\omega, \tilde{g}) = \chi(\omega)\epsilon(\tilde{g})$. If for $\chi \equiv 1$, we have $\epsilon_1 \equiv 1$, we can identify S_1 with the trivial spin structure and $\Gamma(S_1)$ with $C^\infty(G, \Delta_{p,q})$. Let (g, Id) be a global section of $P \cong G \times \mathrm{SO}_+(p,q)$ and let $[\tilde{g}, \mathbf{1}] : U \to Q_\chi$ be a local lift of this section. Then $\gamma(x) = [\{\tilde{g}(x)\}, \mathbf{1}] \in \Gamma(E^{\mathbb{C}}_{|U})$ corresponds to this section and $e_\chi(x) = [\{\tilde{g}(x)\}, \epsilon_\chi(\tilde{g}(x))] = \epsilon_\chi(\tilde{g}(x))\gamma(x)$. Since $\nabla^{E^{\mathbb{C}}}\gamma = 0$, we get

$$\nabla^{E^{\mathbb{C}}_\chi}_V e_\chi = (\mathrm{d}\epsilon_\chi)(V^*)\gamma = \epsilon_\chi^{-1}(\mathrm{d}\epsilon_\chi)(V^*)e_\chi,$$

hence $\omega_\chi(V) = \epsilon_\chi^{-1} V^*(\epsilon_\chi)$, where V^* denotes the lift of $V \in \mathcal{X}(G)$ to Q_χ.

Identifying $\Gamma(S_\chi)$ with $\Gamma(S_1) = C^\infty(G, \Delta_{p,q})$ yields $\nabla^{S_\chi}_V \varphi = \nabla^{S_1}_V \varphi + \epsilon_\chi^{-1} V^*(\epsilon_\chi)\varphi$ for $\varphi \in C^\infty(G, \Delta_{p,q})$.

For instance, consider the Lorentzian torus (T^{1+1}, g_λ). The universal cover is given by $p : \mathbb{R}^2 \to T^{1+1}$, $p(x_1, x_2) = (e^{2\pi \mathrm{i} x_1}, e^{2\pi \mathrm{i} x_2})$. Then $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{R}^2$ by $(z_1, z_2), (x_1, x_2) = (x_1 + z_1, x_2 + z_2)$. On the other hand, $\mathrm{Hom}(\pi_1(T), \mathbb{Z}_2)$ can be identified with $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(a_1, a_2) | a_i \in \{\pm 1\}\}$ by $\chi_1 = \chi(1 \oplus 0) = e^{(\mathrm{i}\pi/2)(1-a_1)}$ and $\chi_2 = \chi(0 \oplus 1) = e^{(\mathrm{i}\pi/2)(1-a_2)}$. We define

$$\epsilon_a(x_1, x_2) = e^{(\mathrm{i}\pi/2)(x_1(1-a_1)+x_2(1-a_2))}.$$

Then $\epsilon_{(1,1)} \equiv 1$ and $\epsilon_\chi((z_1, z_2), (x_1, x_2)) = \chi_1^{z_1}\chi_2^{z_2}\epsilon_\chi(x_1, x_2) = \chi(z_1, z_2)\epsilon_\chi(x_1, x_2)$. Now $\omega_\chi(X_s)(x) = \epsilon_\chi^{-1}((s_1 + s_2)^*)(\epsilon_\chi)(\tilde{x})$, hence

$$\omega_\chi(X_s)(x) = \frac{\mathrm{i}\pi}{2}\left(\frac{1-a_1}{\lambda_1} + \frac{1-a_2}{\lambda_2}\right).$$

Consequently, for a spinor $\tilde{f}u_1 \in \Gamma(S^+_{(a_1,a_2)}) \cong C^\infty(T^{1+1}, \mathbb{C}u_1)$ to be harmonic, we yield the following equation:

$$\nabla^{S_1}_X \tilde{f} + \omega(X)\tilde{f} = \frac{1}{\lambda_1}\partial_{x_1}\tilde{f} + \frac{1}{\lambda_2}\partial_{x_2}\tilde{f} + \frac{\mathrm{i}\pi}{2}\left(\frac{1-a_1}{\lambda_1} + \frac{1-a_2}{\lambda_2}\right)\tilde{f} = 0.$$

Let the development of $\tilde{f}$ into a Fourier series be given by $\tilde{f}(x_1, x_2) = \sum_{k,l\in\mathbb{Z}} \tilde{f}_{kl}\, \mathrm{e}^{2\pi \mathrm{i}(kx_1+lx_2)}$. Then we get the equation:

$$\sum_{k,l\in\mathbb{Z}} \tilde{f}_{kl} \left(\frac{4k - a_1 + 1}{\lambda_1} + \frac{4l - a_2 + 1}{\lambda_2} \right) \mathrm{e}^{2\pi \mathrm{i}(kx_1+lx_2)} = 0.$$

Thus $\tilde{f} \in C^\infty(T^{1+1}, \mathbb{C})$ defines a harmonic spinor if and only if

$$\tilde{f}_{kl} = 0 \ \text{ or } \ 4k - a_1 + 1 = -\frac{\lambda_1}{\lambda_2}(4l - a_2 + 1).$$

Since constants are solutions for the trivial spin structure and $4k - a_1 + 1$ and $4l - a_2 + 1$ are in $\mathbb{Z}$, we finally find

	δ_+ for	
$S_{(a_1,a_2)}$	$(\lambda_1/\lambda_2) \in \mathbb{Q}$	$(\lambda_1/\lambda_2) \notin \mathbb{Q}$
$(+1, +1)$	$+\infty$	1
$(+1, -1)$	$+\infty$	0
$(-1, +1)$	$+\infty$	0
$(-1, -1)$	$+\infty$	0

4.2.2.2. Closed metrics. Next, let $\lambda_i \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}^2}$ be two periodic functions satisfying the additional condition:

$$\partial_{x_2}\lambda_1 + \partial_{x_1}\lambda_2 = 0,$$

that is the form $\lambda = \lambda_1\, \mathrm{d}x_1 - \lambda_2\, \mathrm{d}x_2$ is closed. Therefore we refer to this type of metrics as *closed*. Fix the orthonormal frame $s = (1/\lambda_1\partial_{x_1}, 1/\lambda_2\partial_{x_2})$. Then $\mathrm{div}(X_s) = 0$—a fact we will reconsider later. In terms of Fourier coefficients, the closedness condition may be restated as

$$l\lambda_{1_{kl}} = k\lambda_{2_{kl}}, \tag{4.1}$$

where $\lambda_{i_{kl}}$ denotes the klth Fourier coefficient of λ_i. In particular, we have $\lambda_{1_{0l}} = \lambda_{2_{k0}} = 0$ for l and k different from 0. These formulae will prove useful for the subsequent computations. We fix the trivial spin structure and trivialize with respect to the basis s. Using (3.2), we may rephrase the equation $\nabla^S_{X_s} \tilde{f} u_1 = 0$ as

$$-\lambda_2\partial_{x_1}\tilde{f} = \lambda_1\partial_{x_2}\tilde{f} + \tfrac{1}{2}(\partial_{x_2}\lambda_1 + \partial_{x_1}\lambda_2),$$

so using our additional assumption gives

$$-\lambda_2\partial_{x_1}\tilde{f} = \lambda_1\partial_{x_2}\tilde{f}. \tag{4.2}$$

We remark that the constant spinor u_1 defines a solution that projects onto the torus. Furthermore, if there exists a dense null line, we already know by Proposition 4.5 that there cannot exist any further linearly independent solutions.

In order to find non-trivial solutions we define the function

$$\tilde{f}_\alpha(x_1, x_2) = \exp\left(\mathrm{i}\pi\alpha\left(\int_0^{x_1}(\lambda_1(s, x_2)+\lambda_1(s, 0))\right)\mathrm{d}s - \int_0^{x_2}(\lambda_2(x_1, t) + \lambda_2(0, t))\,\mathrm{d}t\right)$$

for $0 \neq \alpha \in \mathbb{R}$. Then $\tilde{f}_\alpha$ defines a solution for Proposition 4.5 on $\mathbb{R}^2$. In order to be inducible on the torus T, we must impose the double-periodicity of $\tilde{f}_\alpha$, that is $\tilde{f}_\alpha(x_1+n, x_2+m) = \tilde{f}_\alpha(x_1, x_2)$ for $n, m \in \mathbb{Z}$. If $l_i = \iint_{T^{1+1}} \lambda_i \, dT^{1+1}$ denotes the 0th Fourier coefficient of λ_i, we get the following criterion.

Lemma 4.10. $\tilde{f}_\alpha \in C^\infty(\mathbb{R}^2, \mathbb{C})^{\mathbb{Z}^2}$ *if and only if* $\alpha l_i \in \mathbb{Z}$ *for* $i = 1, 2$.

Proof. We have $\tilde{f}_\alpha(x_1+n, x_2+m) = \tilde{f}_\alpha(x_1, x_2)$ if and only if

$$\alpha\left(n\int_0^1 (\lambda_1(s,x_2)+\lambda_1(s,0))\,ds - m\int_0^1 (\lambda_2(x_1,t)+\lambda_2(0,t))\,dt\right) \in 2\mathbb{Z}.$$

Now, $\int_0^1 \lambda_1(s, x_2)\,ds = l_1$ and $\int_0^1 \lambda_2(x_1, t)\,ds = l_2$ (use (4.1)), so $\tilde{f}_\alpha \in C^\infty(\mathbb{R}^2, \mathbb{C})^{\mathbb{Z}^2}$ if and only if $\alpha(nl_1 - ml_2) \in \mathbb{Z}$ for every n and m in $\mathbb{Z}$. □

In particular, if $l_1/l_2 = p/q \in \mathbb{Q}$ (where p and q have no common divisor), then $\tilde{f}_{q/l_2}$ defines a solution. Since the set $\{\tilde{f}_{\alpha m} u_1 | m \in \mathbb{Z}\}$ is linearly independent in $\mathfrak{H}^0_+$, we have $\delta^0_+ = +\infty$.

Example 4.11. Let $c > 0$ be a rational number and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function without zeros with period 1 such that $-f(2x) \neq c$ for every $x \in \mathbb{R}$ and $f(2p_i) = -c/2$ for $p_0 = 0, p_1, \dots, p_n = 1 \in [0, 1]$, but $f'(2p_i) \neq 0$. For instance, we could choose $c = 2$ and $f(x) = (1/10)\cos(2\pi(x + (1/4))) - 1$. Then the diagonal metric defined by $\lambda_1(x_1, x_2) = -f(x_1 - x_2)$ and $\lambda_2 = -f(x_1 - x_2) - c$ is closed. Furthermore, $l_1/l_2 = 1/1 - c \in \mathbb{Q}$, hence $\delta^0_+ = +\infty$. If we express g_λ in the new coordinates (x, y) given by $x = (x_1 - x_2)/2$ and $y = (x_1 + x_2)/2$, we get $g_\lambda(x, y) = (2cf(2x) + c^2)\,dx^2 - 4(f^2(2x) + 2f(2x) + 2)\,dx\,dy + (2cf(2x) + c^2)\,dy^2$. Because of our assumptions $g_\lambda \in \mathcal{G}_2$ (see Proposition 2.1), so g_λ provides an example of a non-conformally flat diagonal metric since it is not complete.

Lemma 4.12. *There are no dense* $\mathcal{X}$*-null lines if and only if* $l_1/l_2 \in \mathbb{Q}$.

Proof. As the global properties of the null lines such as denseness are independent of the parametrization, we can consider the flow of any $\mathcal{X}$-vector field. For instance, we may choose $X = \lambda_2\partial_{x_1} + \lambda_1\partial_{x_2}$, where we assume that $\lambda_1, \lambda_2 > 0$. Let the flow of X be given by $(x_1(t), x_2(t))$. We establish the assertion by computing the rotation number of this flow (see [5] for details). $\lambda_2 > 0$ implies $\lambda_1\,dx_1 - \lambda_2\,dx_2 = 0$. Since the form $\lambda = \lambda_1\,dx_1 - \lambda_2\,dx_2$ is closed, we yield an exact ordinary differential equation on $\mathbb{R}^2$. Hence we have to find an $F : \mathbb{R}^2 \to \mathbb{R}$ such that $\partial_{x_1}F = \lambda_1$ and $\partial_{x_2}F = -\lambda_2$. The initial condition $x_2(0)$ is determined by $F(0, x_2(0)) = c$ for a constant c.

Integration of $\partial_{x_1}F = \lambda_1$ yields $F(x_1, x_2) = \int_0^{x_1} \lambda_1(s, x_2)\,ds + f(x_2)$, where $f'(x_2) = -\lambda_2(x_1, x_2) - \int_0^{x_1}(\partial_2\lambda_1)(s, x_2)\,ds = -\lambda_2(x_1, x_2) + \int_0^{x_1}(\partial_1\lambda_2)(s, x_2)\,ds = -\lambda_2(0, x_2)$. A possible F is given by $F(x_1, x_2) = \int_0^{x_1} \lambda_1(s, x_2)\,ds - \int_0^{x_2} \lambda_2(0, s)\,ds$. We choose $c = 0$ and use the Fourier series of λ_1 and λ_2 to get the following equation:

$$l_1 x_1 + \sum_{\substack{l \\ k\neq 0}} \lambda_{1_{kl}} \frac{e^{2\pi i k x_1} - 1}{2\pi i k} e^{2\pi i l x_2} - l_2 x_2 - \sum_{\substack{l \\ k\neq 0}} \lambda_{2_{kl}} \frac{e^{2\pi i l x_2} - 1}{2\pi i l} = 0.$$

Evaluating in $x_1 = n \in \mathbb{Z}$ yields

$$l_1 n - l_2 x_2(n) - \sum_{\substack{k \\ l\neq 0}} \lambda_{2_{kl}} \frac{e^{2\pi i l x_2} - 1}{2\pi i l} = 0,$$

hence

$$\frac{x_2(n)}{n} = \frac{l_1}{l_2} - \frac{1}{n}\frac{1}{l_2} \sum_{\substack{k \\ l\neq 0}} \lambda_{2_{kl}} \frac{e^{2\pi i l x_2} - 1}{2\pi i l} = 0.$$

Since the Fourier series of smooth functions are absolutely convergent, we have $\rho = \lim_{n\to+\infty}(x_2(n))/n = l_1/l_2$, whence the assertion. □

We finally get the following proposition.

Proposition 4.13. *Let g be conformally equivalent to a closed diagonal metric. Then the following holds*: 1 *and* $+\infty$ *are the only possible values for* δ^0_+. *Furthermore, these dimensions are characterized as follows*:

(i) $\delta^0_+ = 1$ *iff* $l_1/l_2 \notin \mathbb{Q}$ *iff all $\mathcal{X}$-null lines are dense*.
(ii) $\delta^0_+ = +\infty$ *iff* $l_1/l_2 \in \mathbb{Q}$ *iff all $\mathcal{X}$-null lines are closed or asymptotic of a closed $\mathcal{X}$-null line*.

4.3. μ-Surfaces

By Corollary 4.2, a half-spinor which is harmonic or twistor may be seen as an object that is constant along the lifts of the corresponding null lines. Unfortunately, we have no a priori control over the parallel transport in Q, and due to the non-compactness of $\tilde{G}(1,1)$, the lifts of the null lines may even be unbounded (see Proposition 4.17). Therefore, we focus on cases where a direct link between the null lines on (M^{1+1}, g) and the harmonic and twistor spinors can be established, without lifting the null lines to Q.

As we did earlier, we restrict our investigation to the case of positive harmonic spinors. One can prove analogous results by interchanging suitably S^+/S^- and $\mathcal{X}/\mathcal{Y}$, as pointed out above.

Definition 4.14. Let $\varphi \in \Gamma(S^+)$ be a positive harmonic spinor. A smooth function $\mu_\varphi : M^{1+1} \to \mathbb{C}$ verifying

(i) $\mu_\varphi(x) = 0$ if and only if $\varphi(x) = 0$ and
(ii) μ_φ is constant along $\mathcal{X}$-curves

is said to be a mass functional for φ. A μ-surface is a Lorentzian surface admitting a mass functional for every $\varphi \in \Gamma(S^+)$.

We shall give examples of μ-surfaces in the following section (see Example 4.36 and Corollary 4.37 in conjunction with Proposition 4.47). The reason for looking for such mass functionals is the following property.

Lemma 4.15. *Let (M^{1+1}, g) be a μ-surface. Let φ be a positive harmonic spinor and $x \in M$ such that $\varphi(x) = 0$. If $(x_n) \subset l$ for a fixed $\mathcal{X}$-line l converges to x, then $\varphi_{|l} \equiv 0$.*

The general idea to produce obstructions to the inequality $\delta_+ \geq 2$ is to assure that a single zero of a harmonic spinor is propagated along all null lines, therefore forcing the spinor to be zero everywhere. We note the following "heritage principle".

Corollary 4.16. *Let (M^{1+1}, g) be a μ-surface. Let φ be a positive harmonic spinor, and l_∞ a closed $\mathcal{X}$-line. If $\varphi_{|l_\infty} \equiv 0$, then $\varphi_{|l} \equiv 0$ for every asymptotic l of l_∞.*

Although all spin structures on a μ-surface can be treated simultaneously as we shall see, the following proposition illustrates how the non-trivial spin structures differ from the trivial one in terms of the parallel transport.

Proposition 4.17. *Let (M^{1+1}, g) be a μ-surface with a dense $\mathcal{X}$-line l. Then there exists a local section $\tilde{s} : U \to Q$ and a convergent sequence $(x_n) \subset U \cap l$ with $x_n \to x \in U$ such that the following property holds. If $(a_n) \subset \mathbb{R}$ is defined by $\mathcal{P}^Q_{l:x_0 \to x_n}\tilde{s}(x_0) = \tilde{s}(x_n)g_{a_n}$, then for a subsequence (a_{n_l}) we have $a_{n_l} \to \pm\infty$ or $a_{n_l} \to 0$, that is $\{g_{a_n}\}$ is unbounded in $\tilde{G}(1,1)$. Recall that according to Lemma 3.2,*

$$g_{a_n} = \begin{pmatrix} a_n & 0 \\ 0 & 1/a_n \end{pmatrix}.$$

Proof. Assume the opposite. Then consider the horizontal lift l^* of l to Q. Extend l^* to a (continuous) section $\tilde{s}_l : M^{1+1} \to Q$ by $\tilde{s}_l(x) = \lim_n l^*(x_n)$ for $x_n \in l \to x$. This limit exists indeed, since $l^*(x_n) = \mathcal{P}^Q_{l:x_0 \to x_n} l^*(x_0)$ is bounded in Q by assumption. Hence Q would be isomorphic to the trivial spin structure. □

Corollary 4.18. *If there exists a dense $\mathcal{X}$-line on a μ-surface (M^{1+1}, g), then $\delta_+ = 0$ for every non-trivial spin structure.*

Proof. Using the notation of the preceding proposition, we have

$$\begin{aligned} \varphi(x_n) &= [\mathcal{P}^Q_{l:x_0 \to x_n}\tilde{s}(x_0), f_{\tilde{s}}(x_0)u_1] \\ &= [\tilde{s}(x_n), g_{a_n} f_{\tilde{s}}(x_0)u_1] = [\tilde{s}(x_n), f_{\tilde{s}}(x_0)a_n u_1] \to \varphi(x). \end{aligned}$$

Hence, if $a_n \to \pm\infty$, then $f_{\tilde{s}}(x_0) = 0$. If $a_n \to 0$, then $\varphi(x) = 0$. In both cases, the spinor φ has a zero, implying $\varphi \equiv 0$ by Lemma 4.15. □

From now on, we will mostly consider compact μ-surfaces, though the techniques and results can be applied to Lorentzian cylinders as well. Due to the "denseness obstruction" Proposition 4.5, we can restrict our attention to the case where no dense null lines occur. First, we introduce the subsequent notation.

Let $x \in M^{1+1}$ and l_1 and l_2 be two closed $\mathcal{X}$-lines which do not contain x. Since M^{1+1} is homeomorphic to a torus, the connected components of $M^{1+1}\backslash(l_1 \cup l_2)$ are open in M^{1+1} and homeomorphic to a cylinder without boundary. Let $C_{l_1 l_2}(x)$ denote the cylinder which contains x. Its closure is given by $\overline{C_{l_1 l_2}(x)} = C_{l_1 l_2}(x) \cup l_1 \cup l_2$. In the case where $l_1 = l_2$ as a set, we have $\overline{C_{l_1 l_2}(x)} = M^{1+1}$, so the whole torus itself may be considered as a closed cylinder. Now let be x such that l_x is an asymptotic of the two closed null lines l_1 and l_2. Then the cylinder $C_{l_1 l_2}(x)$ will be written $A_{l_1 l_2}(x)$. For further reference, such a cylinder will be called *asymptotic*. Closed null lines are not allowed to be homotopic to a single point, hence there are no more closed null lines in any asymptotic cylinder. Since every asymptotic tends to l_1 or l_2, we get the following lemma.

Lemma 4.19. *Let φ be a positive harmonic spinor on a compact μ-surface. Then its mass functional μ_φ is constant on every closed asymptotic cylinder.*

Thus, if the spinor has a zero in an asymptotic cylinder, it must be zero on the whole cylinder. In order to treat the case where the union of closed null lines is dense in M^{1+1}, we introduce a further type of cylinders which does not contain "ribbons" of closed null lines.

Definition 4.20. A cylinder $C_{l_1 l_2}$ is called non-resonant if for any two arbitrary closed null lines $\tilde{l}_1, \tilde{l}_2$ in the closure of $C_{l_1 l_2}$ there is an asymptotic l in $C_{\tilde{l}_1 \tilde{l}_2}$.

Lemma 4.21. *Let φ be a positive harmonic spinor, and $C = C_{l_1 l_2}$ be a non-resonant cylinder. Then μ_φ is constant on C.*

Proof. Consider the set $A := \{x \in C | l_x \text{ is an asymptotic}\}$. A is open and dense in C. As the total differential of μ_φ vanishes on A as a consequence of Lemma 4.19, the denseness of A implies the result. □

Corollary 4.22. *Let (M^{1+1}, g) be a non-resonant μ-cylinder and let φ_1, φ_2 be two positive harmonic spinors. If there exists a $x \in M^{1+1}$ such that $\varphi_1(x) = \varphi_2(x)$, then $\varphi_1 \equiv \varphi_2$. In particular, every positive harmonic spinor with a zero is identically zero, and $\delta_+ \leq 1$.*

Definition 4.23. Let (M^{1+1}, g) be a Lorentzian surface and l a closed $\mathcal{X}$-line. The spin bundle Q is called $\mathcal{X}$-trivial along l, if the relation $\mathcal{P}^Q_{l:x\to x} q = q$ holds for every $x \in l$ and $q \in (Q_{|l})_x$.

Lemma 4.24. *Let φ be a positive harmonic spinor that has no zero along a closed $\mathcal{X}$-line l. Then Q must be $\mathcal{X}$-trivial along l.*

Proof. By Corollary 4.3, we know that $\varphi(x) = [q, v] = [\mathcal{P}^Q_{l:x\to x} q, v]$. Hence, if $\mathcal{P}^Q_{l:x\to x} q = qg$ for a uniquely determined $g \in \mathrm{Spin}(1, 1)$, we have $g^{-1}v = v$. It follows $g = id$ by Lemma 3.2. □

So far, we have obtained obstructions to the inequality $\delta_+ \geq 2$. In the case where this inequality holds, a third type of cylinder becomes interesting.

Definition 4.25. A closed cylinder $R_{l_1 l_2} = \overline{C_{l_1 l_2}}$ which does not consist of a single $\mathcal{X}$-line is said to be resonant if l_x is closed for every $x \in R_{l_1 l_2}$.

Proposition 4.26. *Let (M^{1+1}, g) be a compact μ-surface.*

(i) *A non-trivial positive harmonic spinor cannot be zero on every resonant cylinder.*
(ii) *If $\delta_+ \geq 2$, then there exists a $\mathcal{X}$-trivial resonant cylinder R on M^{1+1}, that is Q is $\mathcal{X}$-trivial along every closed $\mathcal{X}$-line in R.*

Proof. (ii) is a consequence of (i). To prove the first assertion, let us assume the opposite. It suffices to show that μ_φ is locally constant.

Let $x \in M^{1+1}$. If l_x is an asymptotic, then $\mu_{\varphi|A_{l_1 l_2}(x)}$ is constant by Lemma 4.19. Otherwise, l_x is closed. If for every neighbourhood U of x there exists $x' \in U$ such that l_x is an asymptotic, then x is in the closure of a non-resonant cylinder C. If $x \in \mathrm{int}(C)$, then μ_φ is constant on a neighbourhood of x by Lemma 4.21. If not, then $x \in \partial C \cap \partial R$, where R is a resonant cylinder. Thus $\mu_\varphi \equiv 0$ on a neighbourhood of x, since $\mu_{\varphi|C} \equiv$ const and $\mu_{\varphi|R} \equiv 0$ by assumption. □

On the other hand, whenever there exists an $\mathcal{X}$-trivial resonant cylinder on M^{1+1}, then we can produce harmonic spinors as in Proposition 4.6, since the $\mathcal{X}$-triviality guarantees that the spinors constructed in this way are well defined. Hence we arrive at the following proposition, generalizing the left-invariant case.

Theorem 4.27. *Let (M^{1+1}, g) be a compact μ-surface. Then the only possible dimensions are $\delta_+ = 0, 1$ and $+\infty$. These cases are characterized as follows:*

(i) $\delta_+ \leq 1$ *if and only if either*
- *there exists a dense $\mathcal{X}$-line in which case we have $\delta_+ = 0$ for the non-trivial spin structures or*
- *M^{1+1} is non-resonant or*
- *there exists no $\mathcal{X}$-trivial resonant cylinder on M^{1+1}.*

(ii) $\delta_+ = +\infty$ *if and only if there exists a resonant $\mathcal{X}$-trivial cylinder on M^{1+1}. In this case, we have $\delta_+ = +\infty$ for every spin structure.*

As we have already seen for the left-invariant case, the dimensions $\delta_+ = 0$ and 1 can occur and depend on the given spin structure.

4.4. Spinors and conformal flatness

We will now study the relationship between the existence of harmonic and twistor spinors and conformal flatness. In particular, we will consider the geometric implications of $\mathcal{X}$-triviality.

As we saw in Proposition 2.2, conformal flatness is related to the existence of nowhere vanishing time- resp. spacelike conformal vector fields. With every spinor $\varphi \in \Gamma(S)$, we can canonically associate a vector field that is conformal in the case of a twistor spinor.

Definition 4.28. Let (M^{p+q}, g) be an orientable and time-orientable pseudo-Riemannian spin manifold, and let $\psi \in \Gamma(S)$. We define the associated vector field V_ψ by the equation

$$g(V_\psi, W) = i^{p+1}\langle W \cdot \psi, \psi\rangle$$

for $W \in \mathcal{X}(M^{p+q})$.

A direct computation yields the following proposition (see, for instance [2]).

Proposition 4.29. *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then V_ψ is a conformal vector field. More precisely, we have $\mathcal{L}_{V_\psi} g = (4/n)Re(i^{p+1}\langle D\psi, \psi\rangle)g$.*

Next, we determine the associated vector field of a spinor in signature $(1, 1)$.

Lemma 4.30. *Let (M^{1+1}, g) be time-orientable and $\psi \in \Gamma(S)$. Let $s = (s_1, s_2) : U \to P$ be an orthonormal frame with a lift $\tilde{s}$ to $\tilde{Q}_\xi$ (cf. Section 3). Let $\psi_{\tilde{s}} = \psi^+_{\tilde{s}} u_1 + \psi^-_{\tilde{s}} u_{-1} \in C^\infty(U, \Delta_{1,1})$ be the local trivialization of ψ with respect to $\tilde{s}$. Then $V_\psi = |\psi^+_{\tilde{s}}|^2 X_s - |\psi^-_{\tilde{s}}|^2 Y_s$. In particular, V_ψ is a causal vector field which is timelike if the local components ψ^+_s* and *ψ^-_s have no zeros, and lightlike in case of a half-spinor without zeros.*

Proof. Let w_1 and w_2 be the local components of $W \in \mathcal{X}(M)$ with respect to s, that is $W = w_1 s_1 + w_2 s_2 = [s, w_1 e_1 + w_2 e_2]$. A direct computation yields $< W \cdot \psi, \psi >= (|\psi^-_{\tilde{s}}|^2 + |\psi^+_{\tilde{s}}|^2) w_1 + (|\psi^+_{\tilde{s}}|^2 - |\psi^-_{\tilde{s}}|^2) w_2$. If $V_\psi = V_{\psi 1} s_1 + V_{\psi 2} s_2$, we get $V_{\psi 1} = (|\psi^+_{\tilde{s}}|^2 + |\psi^-_{\tilde{s}}|^2)$ and $V_{\psi 2} = (|\psi^+_{\tilde{s}}|^2 - |\psi^-_{\tilde{s}}|^2)$. Furthermore, since $\lambda = g(X_s, Y_s) > 0$ and $g(V_\psi, V_\psi) = -2\lambda |\psi^+_{\tilde{s}}|^2 |\psi^-_{\tilde{s}}|^2$, the vector field V_ψ is causal. □

Remark 4.31. Let X be a $\mathcal{X}$-vector field and φ a positive harmonic spinor. Then $X \cdot \varphi = 0$. In particular we get $V_\varphi \cdot \varphi = 0$.

Corollary 4.32. *Let (M^{1+1}, g) and $\psi^\pm \in \Gamma(S^\pm)$ be two twistor spinors without zeros. Then (M^{1+1}, g) is conformally flat. In particular if M is compact, then (M^{1+1}, g) is complete.*

Remark 4.33. According to Proposition 4.4, the same result holds for harmonic instead of twistor spinors.

As we have already seen for $\mathcal{X}$-triviality, harmonic resp. twistor spinors without zeros induce certain "flatness" properties. Therefore, we will look more closely to Lorentzian surfaces that admit nowhere vanishing solutions to the harmonic resp. twistor equation.

First, we recall the following statement.

Proposition 4.34. *Let $\varphi \in \Gamma(S)$ be a harmonic spinor on (M^{p+q}, g). Then* $\mathrm{div}(V_\varphi) = 0$.

Definition 4.35. A time-orientable Lorentzian surface (M^{1+1}, g) is said to be $\mathcal{X}$- resp. $\mathcal{Y}$-conformally flat if there exists a global orthonormal frame $s = (s_1, s_2)$ such that $\mathrm{div}(X_s) = 0$ resp. $\mathrm{div}(Y_s) = 0$. We call a Lorentzian surface (M^{1+1}, g) s.c.f. if (M^{1+1}, g) is either $\mathcal{X}$- or $\mathcal{Y}$-conformally flat.

The notion of semi-conformal flatness will be justified in Corollary 4.42.

As we did for spinors, we will concentrate on $\mathcal{X}$-conformally flat surfaces; analogous statements hold for $\mathcal{Y}$-conformally flat ones.

Since every $\mathcal{X}$-vector field can be written as X_s with respect to a suitably chosen basis, a time-orientable Lorentzian surface is $\mathcal{X}$-conformally flat if and only if there exists a $\mathcal{X}$-vector field X such that $\mathrm{div}(X) = 0$. Furthermore, it follows that the notion of semi-conformal flatness is invariant under conformal change of the metric: If there exists a $\mathcal{X}$-vector field X on (M^{1+1}, g) with $\mathrm{div}(X) = 0$, then we can find another $\mathcal{X}$-vector field $\tilde{X} \in \mathcal{X}(M)$ with $\widetilde{\mathrm{div}}(\tilde{X}) = 0$ (where $\widetilde{\mathrm{div}}$ denotes the divergence operator associated with the conformally changed metric $\tilde{g} = \lambda g$) as can be seen from the formula $\widetilde{\mathrm{div}}(V) = V(\ln(\lambda)) + \mathrm{div}(V)$.

Example 4.36.

(i) As we remarked in Section 4.2.2, closed diagonal metrics admit lightlike divergence-free vector fields and are therefore s.c.f.

(ii) We are going to exhibit further examples by a direct computation of the divergence: let us consider a Lorentzian torus with standard coordinates (x_1, x_2) and volume form ω. Let $V = k\partial_{x_1} + l\partial_{x_2}$. Using the formula $\mathrm{d}(i_X\omega) = \mathrm{div}(X)\omega$, we get $\mathrm{div}(V) = \partial_1 k + \partial_2 l + (1/2)V(\ln|\det(g)|)$. In particular, if $\det(g) \equiv 1$, then $\mathrm{div}(V) = \partial_1 k + \partial_2 l$. As semi-conformal flatness is a conformal invariant, we may always assume—by rescaling the metric with the factor $-1/\det(g)$—this assumption to be fulfilled. For instance, if we consider the family of metrics given by (2.1) and (2.2), we get the following corollary.

Corollary 4.37. *Every metric in $\mathcal{G}_1 \cup \mathcal{G}_2$ defines an s.c.f. surface.*

Proposition 4.38. *Let (M^{1+1}, g) be a time-orientable Lorentzian surface. Then the following assertions are equivalent*:

(i) *(M^{1+1}, g) is $\mathcal{X}$-conformally flat.*

(ii) *There exists a global section $M^{1+1} \to P$ such that* $\mathrm{div}(X_s) = 0$.

(iii) *There exists a $\mathcal{X}$-vector field X such that* $\mathrm{div}(X) = 0$.

(iv) $\nabla^{\mathrm{LC}}_{X_s} s_i = 0$ *for* $i = 1, 2$.

(v) *There exists a positive harmonic spinor without zeros*.

(vi) *There exists a negative twistor spinor without zeros*.

Proof. Only the implications (ii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (ii), (v) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (v) need proof. (ii) $\Leftrightarrow$ (v): by a direct application of the Koszul formula, we prove the following lemma. □

Lemma 4.39.

$$-\tfrac{1}{2}g(X_s,[X_s,Y_s]) = \mathrm{div}(X_s) = g(\nabla^{\mathrm{LC}}_{X_s}s_1,s_2) = -g(\nabla^{\mathrm{LC}}_{X_s}s_2,s_1).$$

Then (3.2) implies the following corollary.

Corollary 4.40. *Locally, we have the identity* $\nabla^S_{X_s}[\tilde{s},\varphi_{\tilde{s}}] = [\tilde{s}, X_s(\varphi_{\tilde{s}}) - (1/2)\mathrm{div}(X_s)e_1\cdot e_2\cdot\varphi_{\tilde{s}}]$. *In particular, if* (M^{1+1},g) *is* $\mathcal{X}$*-conformally flat, we get* $\nabla^S_{X_s}[\tilde{s},\varphi_{\tilde{s}}] = [\tilde{s}, X_s(\varphi_{\tilde{s}})]$.

Furthermore, $g(\nabla^{\mathrm{LC}}_{X_s}s_i,s_i) = 0$ for $i,j\in\{1,2\}$. By the lemma, we have $g(\nabla^{\mathrm{LC}}_{X_s}s_1,s_2) = -g(\nabla^{\mathrm{LC}}_{X_s}s_2,s_1) = \mathrm{div}(X_s)$, whence the equivalence.

(v) $\Rightarrow$ (iii): since $\mathrm{div}(e^f X) = e^f(X(f)+\mathrm{div}(X))$, the $\mathcal{X}$-vector field $e^f X$ will be divergence-free if and only if $X(f) = -\mathrm{div}(X)$ holds. By Lemma 4.30, we have $V_\varphi = \lambda X_s$ with $\lambda\neq 0$. Application of Proposition 4.34 yields $X_s(\ln|\lambda|) = -\mathrm{div}(X_s)$.

(ii) $\Rightarrow$ (v): let $\tilde{s}: M^{1+1}\to Q_0$ be a global section in the trivial bundle. Then $\varphi_{\tilde{s}} = u_1$ defines a positive harmonic spinor without zeros.

Corollary 4.41. *On a* $\mathcal{X}$*-conformally flat surface, we have* $\delta^0_+\geq 1$.

Corollary 4.42. (M^{1+1},g) *is conformally flat if and only if* (M^{1+1},g) *is* $\mathcal{X}$*- and* $\mathcal{Y}$*-conformally flat.*

Proof. For a flat metric, every constant defines a harmonic resp. twistor spinor with respect to the trivial spin structure. The implication follows then from Proposition 3.4. We yield the converse from Corollary 4.32. □

Next, we will prove some further properties of $\mathcal{X}$-conformally flat surfaces.

Definition 4.43. A Lorentzian surface (M^{1+1},g) is said to be $\mathcal{X}$- resp. $\mathcal{Y}$-complete, if every $\mathcal{X}$- resp. $\mathcal{Y}$-geodesic is complete. A Lorentzian surface that is $\mathcal{X}$- resp. $\mathcal{Y}$-complete is said to be semi-null complete.

Proposition 4.44. *A compact* $\mathcal{X}$*-conformally flat surface is* $\mathcal{X}$*-complete.*

Proof. By Proposition 4.38 (v), we have $\nabla^{\mathrm{LC}}_{X_s}X_s = 0$, so that the X_s-geodesics are given by the flow of X_s. □

Example 4.45.

(i) The interdependence of semi-conformal flatness, positive harmonic spinors without zeros and semi-completeness is demonstrated by Example 4.11: since the harmonic half-spinors we found have no zeros, the surface must be semi-complete. But we showed that this metric is in $\mathcal{G}_2$, so it is not conformally flat. Hence, from Corollary 4.42 follows that (T^{1+1}, g) is complete for one type of isotropic geodesics, and that there must be incomplete geodesics for the other type, in accordance with Proposition 2.1.

(ii) Consider the following example taken from [8]: let $\tau : [0,1] \to \mathbb{R}$ be a smooth function with $\tau(a) = 0$, but $\tau'(a) \neq 0$, and whose support is strictly contained in $[0, 1]$. Extend τ periodically on the whole real line and define $g^{\tau}_{(x,y)} = 2\,\mathrm{d}x\,\mathrm{d}y - \tau(x)\,\mathrm{d}y^2$. Then $g^{\tau} \in \mathcal{G}'$, hence g^{τ} is lightlike incomplete. For instance, $\gamma(t) = (a, \tau'(a)\ln(t + (1/\tau'(a))))$ is a closed incomplete geodesic which without loss of generality we assume to be $\mathcal{X}$. Thus any positive harmonic spinor must be zero on γ. But as $\tau_{|[-\epsilon,\epsilon]} \equiv 0$ for ϵ sufficiently small, (T^{1+1}, g) contains an $\mathcal{X}$-trivial resonant cylinder, and therefore $\delta_+ = +\infty$. This example shows that there exists not conformally flat tori with $\mathcal{X}$-trivial resonant cylinder which are not $\mathcal{X}$-conformally flat.

Proposition 4.46. *On a $\mathcal{X}$-conformally flat surface, we have $\mathfrak{H}_+ \cong \mathfrak{T}_-$.*

Proof. The maps $\Phi : \mathfrak{H}_+ \to \mathfrak{T}_-, \varphi \mapsto (\mathrm{i}/2)Y_s \cdot \varphi$ and $\Psi : \mathfrak{T}_- \to \mathfrak{H}_+, \psi \mapsto (\mathrm{i}/2)X_s \cdot \varphi$ are bundle isomorphisms inverse of one another. □

Proposition 4.47. *Every s.c.f. surface is a μ-surface.*

Proof. Let $s = (s_1, s_2) : M \to P$ be a (global) orthonormal frame with $\mathrm{div}(X_s) = 0$. We define

$$\mu_\varphi(x) = \langle Y_s \cdot \varphi, \varphi \rangle.$$

Let $\tilde{s}$ be a local lift of s to $\tilde{Q}_{s_1}$. For this section, let $\varphi = [\tilde{s}, \varphi_{\tilde{s}}]$. Then $\mu_\varphi(x) = \langle Y_s \cdot \varphi, \varphi \rangle(x) = -2|\varphi_{\tilde{s}}(x)|^2$, where $|\cdot|$ denotes the absolute value function on $\mathbb{C}$. Thus (i) and (ii) of Definition 4.14 hold.

Since φ is a positive harmonic spinor, we have $\nabla^S_{X_s}\varphi = 0$. Consequently, we get $X_s(\mu_\varphi) = \langle \nabla^{\mathrm{LC}}_{X_s} Y_s \cdot \varphi, \varphi \rangle$. Since $\nabla^{\mathrm{LC}}_{X_s} Y_s = -\nabla^{\mathrm{LC}}_{X_s} s_1 + \nabla^{\mathrm{LC}}_{X_s} s_2 = 0$ by Proposition 4.38 (iv), the assertion follows. □

The classification of the possible dimensions of δ_+ in Theorem 4.27 may be restated as follows.

Theorem 4.48. *Let (M^{1+1}, g) be a compact $\mathcal{X}$-conformally flat Lorentzian surface. Then $\delta_+ = \tau_-$ and the only possible dimensions for δ_+ are $0, 1$ and $+\infty$. These cases are characterized as follows*:

(i) $\delta_+ \leq 1$ *if and only if either*
- *there exists a dense $\mathcal{X}$-line in which case we have $\delta_+ = 0$ for the non-trivial spin structures, or*
- *M^{1+1} is non-resonant, or*
- *there exists no $\mathcal{X}$-trivial resonant cylinder on M^{1+1}.*

Furthermore, $\delta_+^0 = 1$.

(ii) $\delta_+ = +\infty$ *if and only if there exists an* $\mathcal{X}$*-trivial resonant cylinder on* M^{1+1}. *In this case, we have* $\delta_+ = +\infty$ *for every spin structure.*

Next we will show that in some sense $\mathcal{X}$-conformal flatness is forced by positive harmonic spinors which have non-zero "mass".

Definition 4.49. Let (M^{1+1}, g) be a Lorentzian surface and $(P, \pi, M^{1+1}; G)$ a principal fiber bundle over M^{1+1}. A (local) section $s : U \to P$ is said to be $\mathcal{X}$- resp. $\mathcal{Y}$-parallel if for every $\mathcal{X}$- resp. $\mathcal{Y}$-curve $\alpha : [a, b] \to U$, we have $\mathcal{P}^P_{\alpha:a\to b} s(\alpha(a)) = s(\alpha(b))$, where $\mathcal{P}^P_\alpha$ denotes the parallel transport in P along α.

Lemma 4.50. *Let* $s = (s_1, s_2) : U \to P$ *be a local section of the orthonormal frame bundle* P. *Then* s *is* $\mathcal{X}$*-parallel if and only if* $\mathrm{div}(X_s) = 0$. *Furthermore, if* s *can be lifted to a section* $\tilde{s} : U \to Q$ *of* Q, *then* $\tilde{s}$ *is* $\mathcal{X}$*-parallel if and only if* $\mathrm{div}(X_s) = 0$.

Proof. Let α be the flow generated by X_s in U, and let $\mathcal{P}^{\mathrm{LC}}_\alpha$ be the usual parallel transport in TM^{1+1} along α induced by $\mathcal{P}^P$. We have $\nabla^{\mathrm{LC}}_{X_s} s_j(x) = \mathrm{d}/\mathrm{d}t\mathcal{P}^{\mathrm{LC}}_{\alpha:t\to 0} s_j(\alpha(t))_{|t=0} = \mathrm{d}/\mathrm{d}t[s(\alpha(0)), e_j] = 0$, hence $\mathrm{div}(X_s) = 0$ by Proposition 4.38 (v). □

For the converse, let Z denote the Levi–Civita connection in P. Since $\mathrm{div}(X_s) = -(1/2) - g(\nabla^{\mathrm{LC}}_{X_s} s_1, s_2) = 0$, we get $s^*Z(X) = Z(\mathrm{d}s(X)) = 0$ for every $\mathcal{X}$-vector field X (cf. Corollary 4.40). Hence $s^*Z(\alpha'(t)) = 0$ for every $\mathcal{X}$-curve $\alpha : [a, b] \to M^{1+1}$, that is $\alpha^*_{s(\alpha(a))} =$ lift of α starting in $s(\alpha(a)) = s \circ \alpha$. It follows that $\mathcal{P}^P_\alpha s = \alpha^*_{s(\alpha(a))} = s(\alpha(b))$.

Since $f \circ \mathcal{P}^Q = \mathcal{P}^P \circ f$ and $\tilde{s}^*\tilde{Z}(X_s) = -\mathrm{div}(X_s)\omega = 0$ for the lifts of s and Z to Q, we deduce the same result for the spin bundle Q.

The notion of $\mathcal{X}$-triviality can then be reformulated as follows.

Corollary 4.51. *Let* l *be a closed* $\mathcal{X}$*-line. Then* Q *is* $\mathcal{X}$*-trivial along* l *if and only if* l *can be parametrized such that* $\mathrm{div}(l') = 0$. *In particular, such a parametrization makes* l *into a geodesic.*

Proof. Choose an orthonormal frame s such that $l' = s_1 + s_2$ and repeat the reasoning of Lemma 4.50. □

Remark 4.52. As in the case of conformal flatness the condition $\mathrm{div}(X) = 0$ can always be locally realized. Indeed, if $\beta : (a, b) \to U$ is a $\mathcal{Y}$-curve, pick a section $s : |\beta| \to P$ and extend this section on U by parallel transport of $s(\beta(t))$ in the $\mathcal{X}$-direction. Hence $s : U \to P$ is $\mathcal{X}$-parallel by construction and well defined if U is conveniently chosen, that is β intersects every $\mathcal{X}$-line in U only once and there are no closed $\mathcal{X}$-lines (e.g. if U is simply connected). Then $\mathrm{div}(X_s) = 0$ by Lemma 4.50.

4.5. Conclusion

Like for compact Riemannian surfaces, δ_+ depends both on the conformal class of the metric and on the spin structure. δ_+ may be unbounded, in contrast to what is known for the Riemannian case, where the dimension is bounded by $[g + 1/2]$ (g denoting the genus of the surface)—see [6]. Furthermore, for Lorentzian surfaces we have a certain symmetry between harmonic and twistor spinors.

In the case of μ-surfaces, the conformal invariants δ and τ reflect the global behaviour of the null lines. In some regular cases, where the global and local behaviour is quite similar (e.g. for simply connected surfaces or resonant tori), δ and τ are $+\infty$. If a "pathological" behaviour such as dense null lines occurs, then δ and τ are less than or equal 2, and we have a kind of "dynamic" dependence on the conformal class. No intermediate values are attained. Although δ and τ are weaker conformal invariants than the null lines, in some cases they allow us to distinguish between conformal classes. Furthermore, solutions with "mass", that is solutions without zeros, force conformal flatness. All techniques used—above all the characterization of harmonic and twistor half-spinors as a kind of parallel spinors along the lightlike distributions—are genuine for the signature $(1, 1)$. On the other hand, the case of a pseudo-Riemannian signature (p, q) with $p+q \geq 3$ is significantly different. For instance, the dimension of the space of twistor spinors on a connected pseudo-Riemannian manifold is bounded by $2^{[(p+q)/2]+1}$ (see [2,3]).

It is not clear altogether to what extent these techniques can be applied to a general Lorentzian surface or which are the geometric obstructions for doing so. The next obvious step would be to investigate the class of asymptotic cylinders. One could try to find counterexamples of the "heritage principle", that is, harmonic or twistor spinors which are zero on the closed null lines, but have no zeros on the asymptotic cylinder itself, or metrics for which $\delta_\pm$ or $\tau_\pm$ may attain values other than 0, 1 or $+\infty$.

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